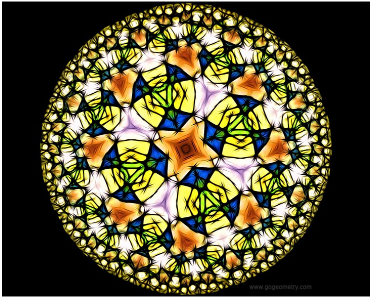
Symplectic geometry of the moduli space of hyperbolic 0-metrics

Eckhard Meinrenken
(based on work with Anton Alekseev)

Groups in action:
in honour of Michèle Vergne
September 5, 2023

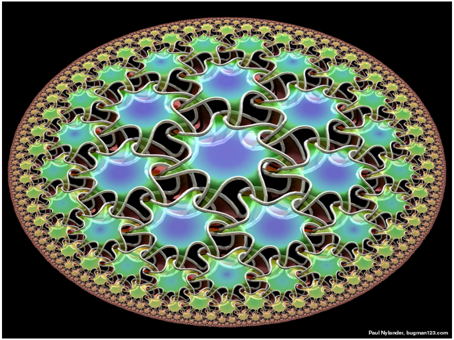


Based on:

A. Alekseev, E.M.: On the coadjoint Virasoro action
(Preprint, arXiv:2211.06216)

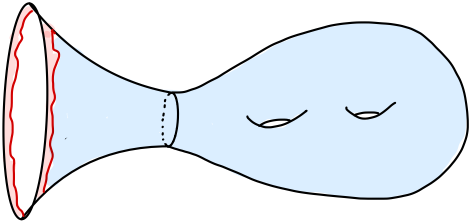
A. Alekseev, E.M.: Symplectic geometry of the moduli space of hyperbolic 0-metrics (in preparation)

1. Motivation



Motivation from physics: JT gravity.

Moduli spaces of Riemann surfaces with wiggly boundary



arising in JT gravity (Maldacena-Stanford-Yang 2016, Saad-Shenker-Stanford 2019, Stanford-Witten 2019, and others).

$\rightsquigarrow$ Schwarzian derivative, Virasoro algebra, DH measures, Mirzakhani recursion formulas etc.

Motivation from physics: JT gravity.

Our take on it:

There is an ∞-dimensional Teichmüller space

$$\mathcal{T}(\Sigma) = \frac{\text{hyperbolic 0-metrics on } \Sigma}{\mathrm{Diff}_\circ(\Sigma, \partial\Sigma)}$$

which is a Hamiltonian space, with momentum map taking values in $\mathrm{vir}_1^*(\partial\Sigma)$.

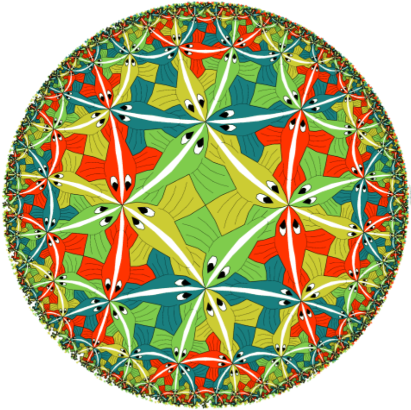
A 0-metric is a metric on the Mazzeo-Melrose 0-tangent bundle

$${}^0T\Sigma \to \Sigma$$

(sections are vector fields vanishing at boundary).

Equivalently, 0-metric means boundary behaviour $\sim \rho^{-2}$.

2. Hyperbolic structures



Hyperbolic structures

Notation.

- $\mathbb{D} \subset \mathbb{C}$ Poincaré disk with standard hyperbolic metric,
- $G = \mathsf{Iso}_+(\mathbb{D})$ isometry group
- $K \subset G$ stabilizer of $(0,0) \in \mathbb{D}$.

Thus

$$G = \mathsf{PSU}(1,1) \cong \mathsf{PSL}(2,\mathbb{R}), \quad K \cong \mathsf{U}(1).$$

Hyperbolic structures: $\partial\Sigma = \emptyset$

Σ compact oriented surface without boundary: $\partial\Sigma = \emptyset$.

Definition

A hyperbolic structure on Σ is an atlas with charts $\phi_\alpha \colon U_\alpha \to \mathbb{D}$, transition maps in G. Let

$$\mathrm{Hyp}(\Sigma) = \{\text{hyperbolic structures on } \Sigma\}$$

and define Teichmüller space

$$\mathrm{Teich}(\Sigma) = \mathrm{Hyp}(\Sigma) / \mathrm{Diff}_o(\Sigma).$$

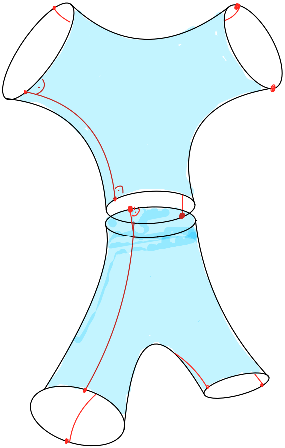
Hyperbolic structure $\Leftrightarrow$ hyperbolic metric g (i.e., $K_g = -1$).

Hyperbolic structures: $\partial\Sigma = \emptyset$

$\text{Teich}(\Sigma) \cong (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}$ by Fenchel-Nielsen length and twist parameters

$$\ell_1, t_1, \ldots, \ell_{3g-3}, t_{3g-3}$$

from pants decomposition



Hyperbolic structures: $\partial\Sigma = \emptyset$

- Weil (1958), Ahlfors (1961): Teich(Σ) has a canonical Kähler structure.
- Wolpert (1981): In Fenchel-Nielsen coordinates,

$$\omega = \sum_{i=1}^{3g-3} \mathrm{d}\ell_i \wedge \mathrm{d}t_i.$$

- Goldman, Hitchin (1987): Teich(Σ) is a connected component of

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G,$$

with Atiyah-Bott symplectic structure.

Hyperbolic structures: $\partial\Sigma \neq \emptyset$

Question: How to define $\mathrm{Hyp}(\Sigma)$ if Σ has boundary?

Answer: Replace $\mathbb{D}$ with $\overline{\mathbb{D}}$.

Hyperbolic structures: $\partial\Sigma \neq \emptyset$

Σ compact oriented surface (possibly) with boundary $\partial\Sigma \neq \emptyset$.

Definition

A hyperbolic structure on Σ is an atlas with charts $\phi_\alpha \colon U_\alpha \to \overline{\mathbb{D}}$, transition maps in G. Let

$$\mathrm{Hyp}(\Sigma) = \{\text{hyperbolic structures on } \Sigma\}$$

and define Teichmüller space

$$\mathrm{Teich}(\Sigma) = \mathrm{Hyp}(\Sigma)/\ {}^0\,\mathrm{Diff}_o(\Sigma).$$

Hyperbolic structure $\Leftrightarrow$ hyperbolic 0-metric g.

Hyperbolic structures: $\partial\Sigma \neq \emptyset$

Example

For $\Sigma = \overline{\mathbb{D}}$, with standard 0-metric

$$g = 4\frac{dr^2 + r^2 d\theta^2}{(1-r^2)^2}$$

we obtain (a version of) universal Teichmüller space

$$\mathrm{Teich}(\overline{\mathbb{D}}) = \mathrm{Diff}_+(\partial\mathbb{D})/G.$$

Example

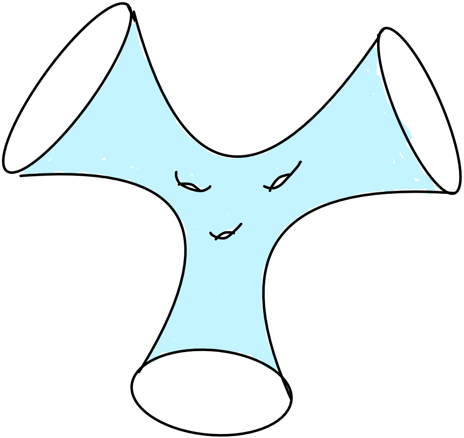
For cylinder $\Sigma = [-1,1] \times S^1$, obtain

$$\mathrm{Teich}(\Sigma) = (\widetilde{\mathrm{Diff}}_+(S^1) \times \widetilde{\mathrm{Diff}}_+(S^1))/\mathbb{R}.$$

Example

For $\chi(\Sigma) = 2 - 2g - r < 0$, have Fenchel-Nielsen parameters

$$\mathrm{Teich}(\Sigma) = (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+r} \times \prod_{i=1}^{r} (\mathbb{R}_{>0} \times \widetilde{\mathrm{Diff}}_+(S^1)).$$



3. Symplectic structure

Symplectic structure on $\mathrm{Teich}(\Sigma)$ for $\partial\Sigma = \emptyset$

Back to case $\partial\Sigma = \emptyset$. Principal G-bundles $P \to \Sigma$ classified by

$$\mathrm{e}(P) \in \pi_1(G) = \mathbb{Z}.$$

Fix

$$\begin{array}{ccc} P & \xrightarrow[\sigma]{} & \mathbb{D} \\ {\scriptstyle \pi}\big\downarrow & & \\ \Sigma & & \end{array}$$

where

- $\pi \colon P \to \Sigma$ principal bundle with $\mathrm{e}(P) = \chi(\Sigma)$
- $\sigma \colon P \to \mathbb{D}$ a G-map.

Since $\mathbb{D} = G/K$, choice of σ gives reduction of structure group $P_K = \sigma^{-1}(0)$.The choice of (P, σ) is unique up to isomorphism.

Symplectic structure on $\mathrm{Teich}(\Sigma)$ for $\partial\Sigma = \emptyset$

$$\begin{array}{ccc} P & \xrightarrow{\sigma} & \mathbb{D} \\ \downarrow{\scriptstyle \pi} & & \\ \Sigma & & \end{array}$$

Definition

$\theta \in \mathcal{A}(P)$ is positive if the composition

$$\pi^* T\Sigma \xrightarrow{j^\theta} TP \xrightarrow{T\sigma} T\mathbb{D}$$

is an oriented isomorphism.

In this case, $T\Sigma$ inherits a metric.

In local trivialization $(P_K|_U \cong U \times K)$, θ has connection 1-form

$$A = \frac{1}{2} \begin{pmatrix} \alpha_2 & \alpha_1 - \kappa \\ \alpha_1 + \kappa & -\alpha_2 \end{pmatrix}$$

Positivity means that α_1, α_2 is oriented coframe.

Symplectic structure on $\mathrm{Teich}(\Sigma)$ for $\partial\Sigma = \emptyset$

Proposition

1. *If $\theta \in \mathcal{A}^{\mathrm{pos}}(P)$ is flat, then g is hyperbolic.*
2. *Every hyperbolic metric g arises from flat, positive connection, unique up to $\mathrm{Gau}(P, \sigma)$.*

Thus:

$$\mathrm{Hyp}(\Sigma) = \mathcal{A}^{\mathrm{pos}}_{\mathrm{flat}}(P) / \mathrm{Gau}(P, \sigma)$$

and consequently

$$\mathrm{Teich}(\Sigma) = \mathcal{A}^{\mathrm{pos}}_{\mathrm{flat}}(P) / \mathrm{Aut}_{\mathrm{o}}(P, \sigma)$$

where $\mathrm{Aut}_{\mathrm{o}}(P, \sigma)$ are automorphisms preserving σ and with base map in $\mathrm{Diff}_{\mathrm{o}}(\Sigma)$.

This is a symplectic quotient.

Symplectic structure on Teich(Σ) for $\partial\Sigma = \emptyset$

Recall Atiyah-Bott form on $\mathcal{A}(P)$

$$\omega(a, b) = \int_\Sigma a \dot{\wedge} b,$$

for $a, b \in T_\theta\mathcal{A}(P) = \Omega^1(\Sigma, \mathfrak{g}(P))$. We have:

$$\mathrm{Teich}(\Sigma) = \mathcal{A}^{\mathrm{pos}}(P) /\!\!/ \mathrm{Aut}_\circ(P, \sigma).$$

One can verify that this symplectic form on Teich(Σ) gives Wolpert's formula.

Symplectic structure on Teich(Σ) for $\partial\Sigma \neq \emptyset$

Question: How to define symplectic structure if Σ has boundary?

Answer: Replace $\mathbb{D}$ with $\overline{\mathbb{D}}$.

Symplectic structure on Teich(Σ) for $\partial\Sigma \neq \emptyset$

For Σ with boundary, consider

$$\begin{array}{ccc} P & \xrightarrow[\sigma]{} & \overline{\mathbb{D}} \\ {\scriptstyle\pi}\downarrow & & \\ \Sigma & & \end{array}$$

- $\pi\colon P \to \Sigma$ a G-bundle,
- $\sigma\colon P \to \overline{\mathbb{D}}$ a G-equivariant morphism of manifolds with boundary.

- Over interior, get reduction of structure group to $K \cong \mathrm{U}(1)$

$$P_K = \sigma^{-1}(0) \subset P|_{\mathrm{int}(\Sigma)}$$

- At boundary, get reduction of structure group to $B \cong \mathbb{R} \rtimes \mathbb{R}_{>0}$

$$P_B = \sigma^{-1}(i) \subset P|_{\partial\Sigma}$$

B contractible $\rightsquigarrow$ trivializations of $P|_{\partial\Sigma}$ $\rightsquigarrow$ $\mathrm{e}(P,\sigma) \in \pi_1(G) = \mathbb{Z}$.

Symplectic structure on $\text{Teich}(\Sigma)$ for $\partial\Sigma \neq \emptyset$

Fix (P, σ) with $\mathsf{e}(P, \sigma) = \chi(\Sigma)$.

Definition

$\theta \in \mathcal{A}(P)$ is positive if the composition

$$\pi^* T\Sigma \xrightarrow{j^\theta} TP \xrightarrow{T\sigma} T\overline{\mathbb{D}}$$

is an oriented isomorphism.

In this case, $T\Sigma$ inherits a 0-metric from that on $T\overline{\mathbb{D}}$.

Symplectic structure on $\mathrm{Teich}(\Sigma)$ for $\partial\Sigma \neq \emptyset$

Proposition

Let $\theta \in \mathcal{A}^{\mathrm{pos}}(P)$ be a positive connection.

1. *If θ is flat, then g is a hyperbolic 0-metric.*
2. *Every hyperbolic 0-metric g arises from flat, positive connection, unique up to $\mathrm{Gau}(P, \sigma)$.*

Thus:

$$\mathrm{Hyp}(\Sigma) = \mathcal{A}^{\mathrm{pos}}_{\mathrm{flat}}(P) / \mathrm{Gau}(P, \sigma)$$

and consequently

$$\mathrm{Teich}(\Sigma) = \mathcal{A}^{\mathrm{pos}}_{\mathrm{flat}}(P) / \,{}^0\mathrm{Aut}_{\mathrm{o}}(P, \sigma)$$

where ${}^0\mathrm{Aut}_{\mathrm{o}}(P, \sigma)$ are automorphisms preserving σ and with base map in ${}^0\mathrm{Diff}_{\mathrm{o}}(\Sigma)$.

This is not (quite) a symplectic quotient

Symplectic structure on $\mathrm{Teich}(\Sigma)$ for $\partial\Sigma \neq \emptyset$

Let $\partial\sigma\colon \partial P \to \partial\mathbb{D}$ restriction to the boundary. Have surjective map

$$ {}^0\mathrm{Aut}_o(P,\sigma) \to \mathrm{Gau}(\partial P, \partial\sigma). $$

Let ${}^0\mathrm{Aut}_o(P,\partial P,\sigma)$ be its kernel, and put

$$ \widehat{\mathrm{Teich}}(\Sigma) = \mathcal{A}^{\mathrm{pos}}_{\mathrm{flat}}(P) / {}^0\mathrm{Aut}_o(P,\partial P,\sigma) $$

Lemma

This is a symplectic quotient:

$$ \widehat{\mathrm{Teich}}(\Sigma) = \mathcal{A}^{\mathrm{pos}}(P) /\!\!/ {}^0\mathrm{Aut}_o(P,\partial P,\sigma) $$

Symplectic structure on $\text{Teich}(\Sigma)$ for $\partial\Sigma \neq \emptyset$

Lemma

The residual action of $\text{Gau}(\partial P, \partial\sigma)$ *on* $\widehat{\text{Teich}}(\Sigma)$ *has an affine moment map taking values in a single coadjoint orbit*

$$\mathcal{O} \subset \widehat{\mathfrak{gau}}_1^*(\partial P, \partial\sigma).$$

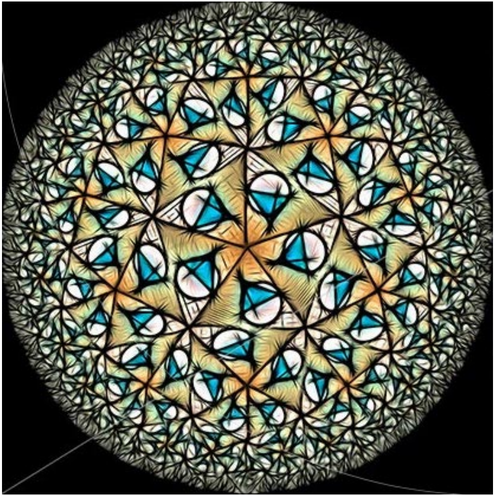
Therefore:

$$\text{Teich}(\Sigma) = \big(\widehat{\text{Teich}}(\Sigma) \times \mathcal{O}^-\big) /\!\!/ \, \text{Gau}(\partial P, \partial\sigma)$$

acquires a symplectic structure.

Next, we want to show it's a Hamiltonian Virasoro space.

4. Moment map



Central extensions

Principle: Given an affine action

$$H \circlearrowright \mathcal{E}$$

with underlying linear action $\mathrm{Ad}\colon H \circlearrowright \mathfrak{h}^*$, get central extension

$$0 \to \mathbb{R} \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0$$

such that $\mathcal{E} \cong \widehat{\mathfrak{h}}^*_1$, the affine hyperplane at level 1.

In fact, $\widehat{\mathfrak{h}} = \mathrm{Hom}_{\mathrm{aff}}(\mathcal{E}, \mathbb{R})$, $\quad [\widehat{X}, \widehat{Y}](\mu) = \langle X, Y \cdot \mu \rangle$.

Example

$P \to C$ principal G-bundle over oriented circle, $\mathrm{Gau}(P) \circlearrowright \mathcal{A}(P)$ defines $\widehat{\mathfrak{gau}}(P)$.

Virasoro Lie algebra

Here $H = \mathrm{Diff}_+(\mathsf{C})$ with C = oriented circle. Have

$$\mathfrak{h} = \mathrm{Vect}(\mathsf{C}) = |\Omega|_{\mathsf{C}}^{-1}, \quad \mathfrak{h}^* = |\Omega|_{\mathsf{C}}^{2} \quad \text{(quadratic differentials)}$$

Definition

A Hill operator is a 2nd order linear differential operator

$$L\colon |\Omega|_{\mathsf{C}}^{-\frac{1}{2}} \to |\Omega|_{\mathsf{C}}^{\frac{3}{2}}$$

such that $L^* = L, \quad \sigma(L) = 1.$

$\mathrm{Diff}_+(\mathsf{C}) \circlearrowright \mathrm{Hill}(\mathsf{C})$; linear action the coadjoint action on $\mathrm{Vect}(\mathsf{C})^*$.

Definition

$\mathrm{vir}(\mathsf{C}) = \widehat{\mathrm{Vect}}(\mathsf{C})$ is the Virasoro Lie algebra.

Virasoro Lie algebra

In coordinates $C = S^1$:

- Hill operators:

$$Lu = u'' + \mathcal{T}u,$$

with Hill potential $\mathcal{T} \in |\Omega|^2_{S^1} \cong C^\infty(S^1)$.

- $\mathrm{Diff}_+(S^1)$-action

$$(\mathsf{F}^{-1} \cdot \mathcal{T}) = (F')^2\ (F^*\mathcal{T}) + \tfrac{1}{2}\mathcal{S}(\mathsf{F}).$$

Here

$$\mathcal{S}(\mathsf{F}) = \frac{\mathsf{F}'''}{\mathsf{F}'} - \frac{3}{2}\Big(\frac{\mathsf{F}''}{\mathsf{F}'}\Big)^2.$$

is the Schwarzian derivative.

The moment map

The moment map

$$\Phi\colon \text{Teich}(\Sigma) \to \text{Hill}(\partial\Sigma), \quad [g] \mapsto \Phi([g]) = \mathcal{T}$$

has the following description. Choose adapted coordinates x, y near boundary, write

$$\text{dvol} = \frac{a(x)}{y^2} \text{d}x \wedge \text{d}y + \dots .$$

Let $\kappa(x, y)$ be the geodesic curvature of $t \mapsto (x + t, y)$.

Theorem (Alekseev-M)

$$\mathcal{T}(x) = \tfrac{1}{2}\Big(\frac{a''}{a} - \frac{3}{2}\big(\frac{a'}{a}\big)^2\Big) + \frac{a^2}{2} \lim_{y\to 0} \frac{\kappa(x,y) - 1}{y^2}.$$

(Cf. Maldacena-Stanford-Yang 2016.)

5. Conclusion

JOUYEUX ANNIVERSAIRE, MICHÈLE!

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